Analysis I Lecture 17

Table of limits

$$\lim_{\boxed{}} f(x) \boxed{= } \iff \boxed{}\ \boxed{\phantom{\exists N>0 \text{ s.t. }}}\ \boxed{}$$

$$\lim_{x\to x_0} f(x) = +\infty \iff \forall M>0 \ \exists N>0 \text{ s.t. } \forall x>N$$

choose finite limit or $\pm\infty$

$f(x) > M$

choose it x_0 finite

$\pm\infty$

Table of limits

$$\lim_{\boxed{}} f(x) \boxed{= } \iff \boxed{} \ \boxed{} \ \boxed{\rightarrow}$$

• $x \to x_0$	• $\exists\, \delta > 0$ s.t. $\forall x \in (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$
• $x \to x_0^+$	• $\exists\, \delta > 0$ s.t. $\forall x \in (x_0, x_0 + \delta)$
• $x \to x_0^-$	• $\exists\, \delta > 0$ s.t. $\forall x \in (x_0 - \delta, x_0)$
• $x \to +\infty$	• $\exists\, N > 0$ s.t. $\forall x > N$
• $x \to -\infty$	• $\exists\, N > 0$ s.t. $\forall x < -N$

$= \ell \in \mathbb{R}$	$\forall \varepsilon > 0$	$\lvert f(x) - \ell \rvert < \varepsilon$
$= +\infty$	$\forall M > 0$	$f(x) > M$
$= -\infty$	$\forall M > 0$	$f(x) < -M$

Left and right continuity

Definition Let $f : E \to \mathbb{R}$ and $x_0 \in E$

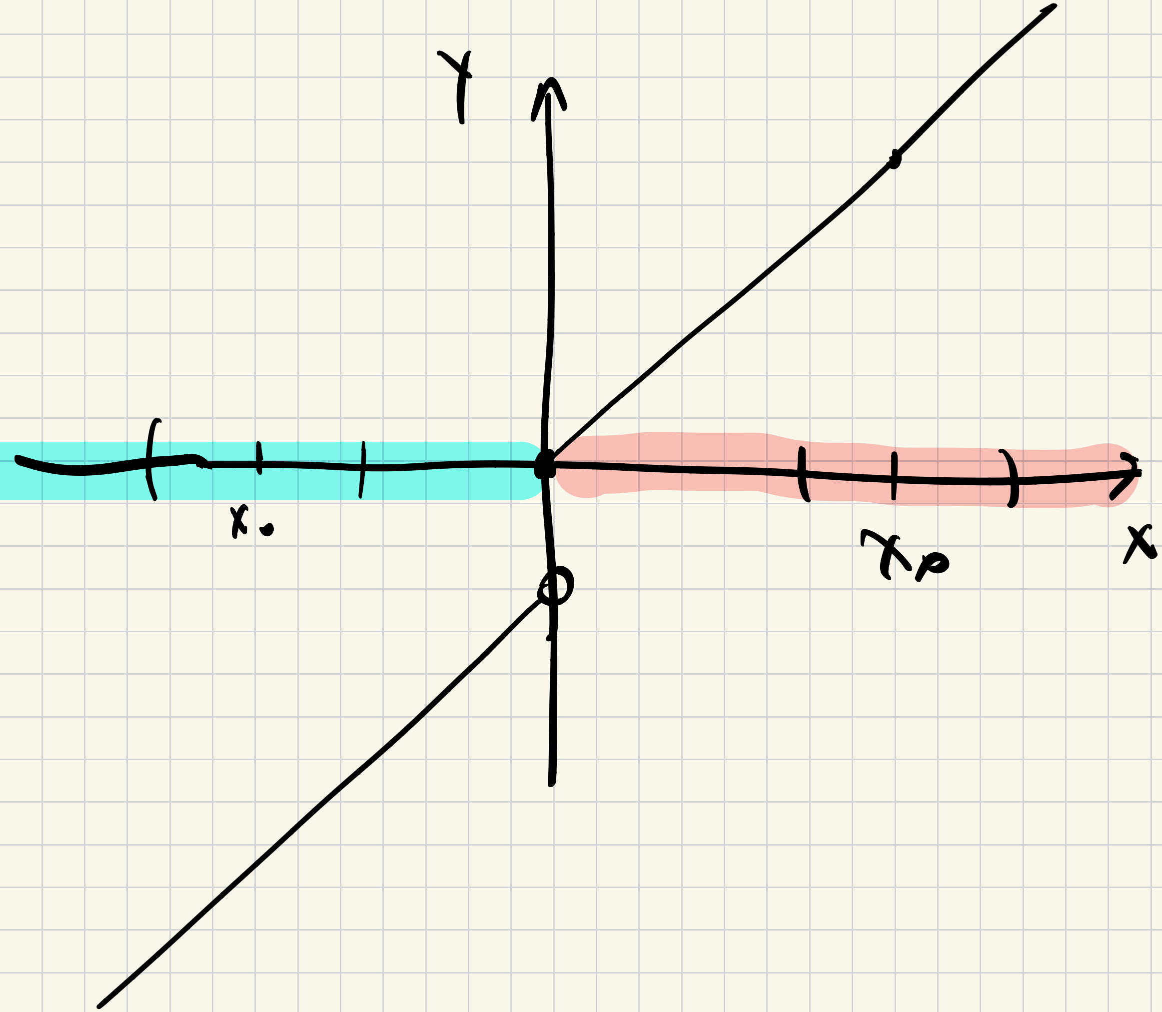
we say that

$f(x)$ is Left continuous at x_0 if $\lim\limits_{x \to x_0^-} f(x) = f(x_0)$

$f(x)$ is Right continuous at x_0 if $\lim\limits_{x \to x_0^+} f(x) = f(x_0)$

Example

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ x-1 & \text{if } x < 0 \end{cases}$$



- $f(x)$ is continuous for every $x_0 \neq 0$
- $\lim_{x \to 0} f(x)$ does not exists

$\Rightarrow$ discontinuous at 0.

• But we have

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (x-1) = -1$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x = 0$$

$$\lim_{x \to 0^-} f(x) \neq f(0) = 0 = \lim_{x \to 0^+} f(x)$$

$\Rightarrow$ f is right continuous but not left continuous at 0.

<u>Definition</u> 1) We say that $f(x)$ is continuous on $[a,b]$ if

- f is continuous at every point $x_0 \in (a,b)$
- f is right continuous at a
- f is left continuous at b

<u>Similarly</u>

2) $f: [a, b) \to \mathbb{R}$ is cont. if

- f is cont. on (a, b)
- f is right-cont at a

3) $f: (a, b] \longrightarrow \mathbb{R}$ is cont. if

- f is cont. on (a, b)
- f is left-cont. at b.

Similarly

2) $f : [a, +\infty) \to \mathbb{R}$ is cont. if

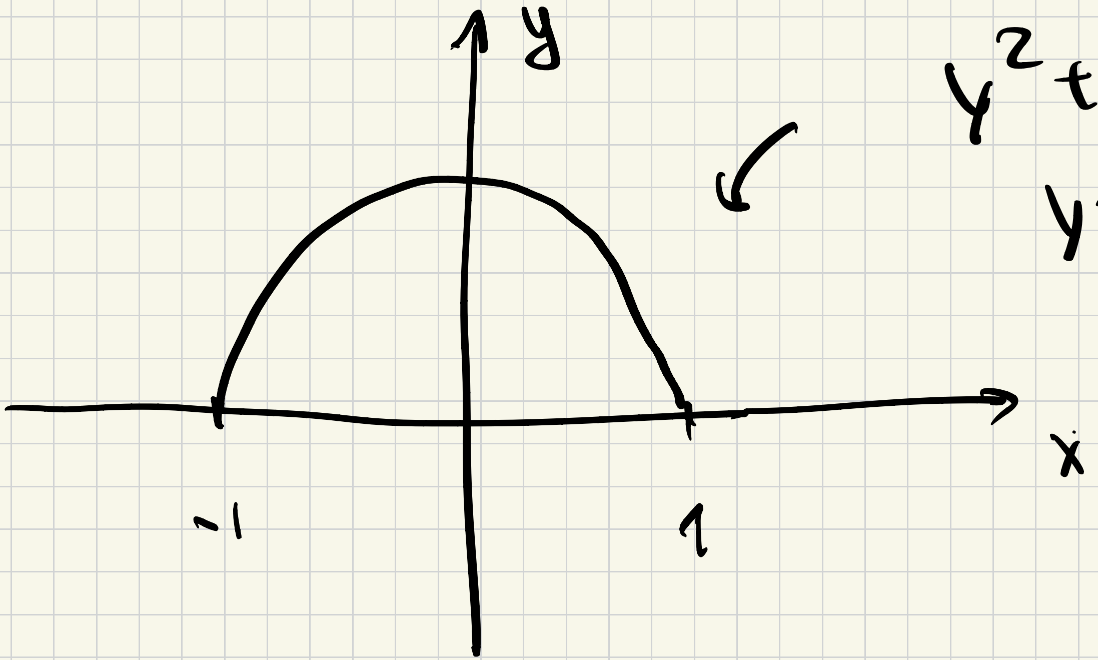
- f is cont. on $(a, +\infty)$
- f is right-cont at a

3) $f : (-\infty, b] \longrightarrow \mathbb{R}$ is cont. if

- f is cont. on $(-\infty, b)$
- f is left-cont. at b.

Example $f : [-1, 1] \to \mathbb{R}$ defined by

$$f(x) = \sqrt{1 - x^2}$$



$y^2 + x^2 = 1$

$y \geq 0$

$\sqrt{1-x^2}$ is continuous on $[-1, 1]$

1) f is cont. on $(-1, 1)$

Later today

2) $\lim\limits_{x \to -1^+} \sqrt{1-x^2} = \underline{\underline{0}} = f(-1)$

Use change of variables and fact $\lim\limits_{x \to 0^+} \sqrt{x} = 0$.

3) $\lim\limits_{x \to 1^-} \sqrt{1-x^2} = 0 = f(1)$.

So we get that

f is continuous on $[-1, 1]$.

Operations preserving continuity.

<u>Proposition</u> Let $f, g : D \to \mathbb{R}$ and $x_0 \in D$ be (left/right) continuous at x_0

1) $f \pm g$ is (left/right) continuous at x_0
2) $f \cdot g$ is (left/right) continuous at x_0
3) if $g(x_0) \neq 0$ then f/g is (left/right) continuous at x_0.

Recall

Change of variables:

If f is cont. at x_0

g is cont. at $f(x_0)$

then $g \circ f$ is cont. at x_0.

List of continuous functions:

- Polynomials : easy corollary since polynomials obtained from $f(x) = x$ by arithmetic operations.
- trigonometric $\left(\sin(x),\ \cos(x),\ \tan(x),\ \cot(x) \right)$
- Exponentials
- logarithms
- roots

E.g. • $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$ continuous

since e^x is cont

e^{-x} is cont

• $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$ is continuous

• $\sqrt{\cos^2(e^x) + 5\log_2(x)^5}$ is cont.

Extension by continuity

Definition / Proposition Let $f: D \to \mathbb{R}$ continuous function and let $x_0 \notin D$ s.t. $\lim_{x \to x_0} f(x)$ exists. Define

$$\tilde{f}_{x_0}(x) = \begin{cases} f(x) & \text{if } x \in D \\ \lim_{x \to x_0} f(x) & \text{if } x = x_0. \end{cases}$$

Extension of f to x_0

Then $\tilde{f}_{x_0}$ is continuous.

Example $f(x) = \frac{\tan(x)}{x}$ for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus \{0\}$

In fact we can extend $f(x)$ to 0:

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\frac{\sin(x)}{\cos(x)}}{x} = \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{1}{\cos(x)}$$

$$= \underbrace{\lim_{x \to 0} \frac{\sin x}{x}}_{=1} \cdot \underbrace{\lim_{x \to 0} \frac{1}{\cos(x)}}_{=1} = 1$$

So the extension of $\frac{\tan(x)}{x}$ to 0

is given by

$$\tilde{f}_0(x) = \begin{cases} \frac{\tan(x)}{x} & \text{for } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\setminus\{0\} \\ 1 & \text{if } x = 0 \end{cases}$$

is continuous on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Note:

$$\lim_{x \to -\frac{\pi}{2}^+} \frac{\tan(x)}{x} = +\infty$$

So we cannot extend f to $-\frac{\pi}{2}$.

Continuous functions on intervals.

(a,b) or $[a,b]$ or $(a,b]$ or $[a,+\infty)$ ---

Notation Let "I" be an interval

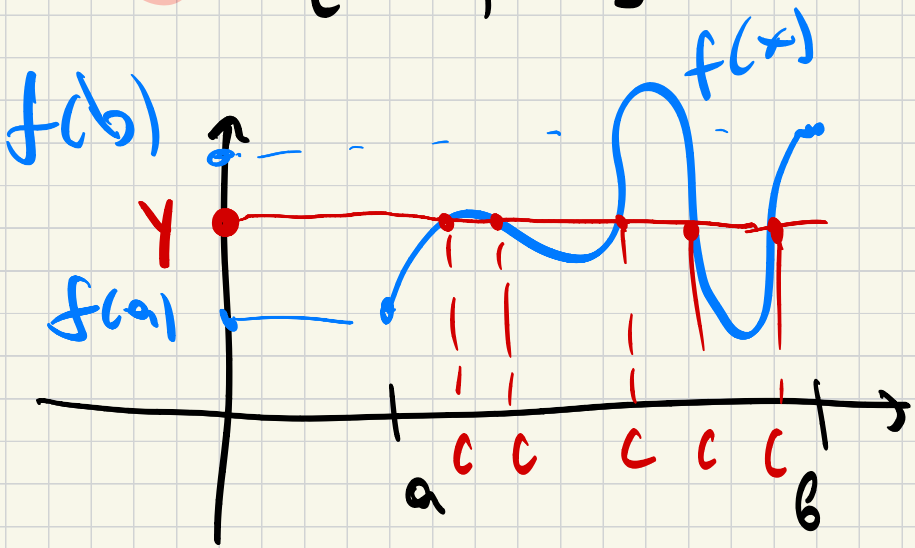
$$C^0(I) := \{ f : I \to \mathbb{R} \mid f \text{ is continuous on } I \}.$$

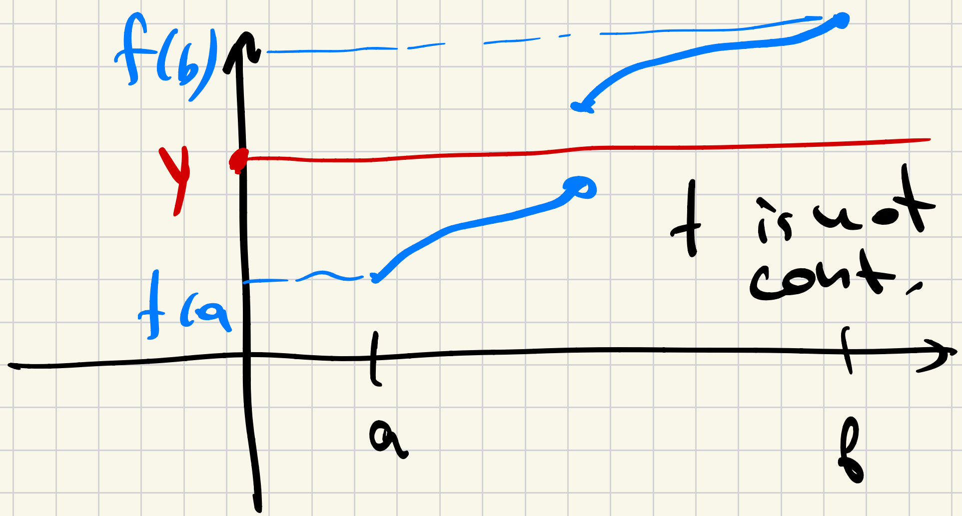
We say f is C^0 instead of f is continuous.

Theorem (Intermideate value theorem)

Theorem 1.66 in Notes on functions.

Let I be an interval, $f \in C^0(I)$ and $a, b \in I$ with $a < b$. Then for every y between $f(a)$ and $f(b)$, there exists $c \in [a, b]$ s.t. $f(c) = y$.





Corollary Every odd degree polynomial with real coeff. has a root.

Proof by intermediate value thm:

$P(x) = x^{2n+1} + \dots + a_1 x + a_0$ then

1st $P(x)$ is continuous.

2nd • if $x \ll 0$ then $P(x) < 0$

very small

since $x^{2n+1} < 0$ and

this term will dominate

the rest of $P(x)$.

• if $x \gg 0$ the $P(x) > 0$

$\Rightarrow$ $\exists c$ s.t. $P(c) = 0$.

Example $f(x) = e^x - 5\cos(x^2)$

We can use IVT to show that

$f(x) = 0$ has a solution:

Indeed

$$f(0) = 1 - 5\cos(0) = -4 < 0$$

$$f(3) = \underbrace{e^3}_{\substack{>8 \\ \text{since } e>2}} - \underbrace{5\cos(9)}_{\leq 5} > 0$$

since $f(x) = e^x - 5\cos(x^2)$ is

continuous

and $f(0) < 0$, $f(3) > 0$

$\exists\ 0 \leq c \leq 3$ s.t. $f(c) = 0$ by

IVT.

Intermediate Value Theorem.

Corollary (Banach fixed point theorem)
Corollary 1.69 in Notes on

Let $f:[a,b]\to[a,b]$ be continuous function

then $\exists\, c\in[a,b]$ s.t. $f(c)=c$.

Proof: Notice that $f(x)=x$ iff

$$f(x)-x=0.$$

So it is enough to show that $(f(x)-x)$ has a root in $[a,b]$.

<u>1st</u> $g(x)=f(x)-x$ is cont. on $[a,b]$

$g(a) = f(a)-a \geq 0$ since $f(a)\in[a,b]$

$g(b) = f(b)-b \leq 0$ since $f(b)\in[a,b]$

$\Rightarrow$ 0 is between $(f(a)-a)$ and

$(f(b)-b)$ so by

IVT $\exists\, c\in[a,b]$ s.t $g(c)=0$ $\blacksquare$

Maximum / Minimum

Definition We say that $f : D \to \mathbb{R}$ admits a maximum (minimum) if $\exists\, x_0 \in D$ s.t.

$$f(x_0) \geq f(x) \quad \left(f(x_0) \leq f(x)\right) \quad \forall\, x \in D.$$

In this case we we write $\max_{x \in D} f(x) = f(x_0)$

$$\left(\min_{x \in D} f(x) = f(x_0) \right).$$

Theorem

Let I be an interval and $\underbrace{f \in C^0(I)}_{f \text{ is continuous on } I \;=}$. Then,

1) The image of f, $f(I) = \mathrm{Im}(f)$ is an interval.

2) If f is strictly monotone I is an open interval then $f(I)$ is an open interval Cor. 1.72

3) if $I = [a, b]$ is closed then $\min\limits_{x \in I} f$, $\max\limits_{x \in I} f$ exist and

$$f(I) = \left[\min_{x \in I} f(x), \max_{x \in I} f(x)\right].$$

Thm 1.63

Examples